

# Quasi-biharmonic Lagrangian surfaces in Lorentzian complex space forms \*

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## Abstract

In this paper, we introduce the notion of a quasi-biharmonic submanifold in a pseudo-Riemannian manifold and classify quasi-biharmonic marginally trapped Lagrangian surfaces in Lorentzian complex space forms.

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## 1 Introduction

A submanifold with lightlike mean curvature vector field is called a *marginally trapped* submanifold. In the theory of cosmic black holes, a marginally trapped surface in a space-time plays an extremely important role. Recently, some classification results on marginally trapped surfaces from the viewpoint of differential geometry have been obtained (see, for instance, [5]).

On the other hand, a submanifold is called *biharmonic* if the bitension field of the isometric immersion defining the submanifold vanishes identically. The theory of biharmonic submanifolds has advanced greatly during this last decade (see, for instance, [1] and [7]). This paper introduces the notion of a *quasi-biharmonic* submanifold. It is a submanifold such that the bitension field of the isometric immersion defining the submanifold is lightlike at each point.

In [9], the author has classified biharmonic marginally trapped Lagrangian surfaces in Lorentzian complex space forms. They exist only in the flat Lorentzian complex plane. In this paper, we classify quasi-biharmonic marginally trapped Lagrangian surfaces in Lorentzian complex space forms. We find that the situation in the quasi-biharmonic case is quite different from the biharmonic case. In fact, there exist a lot of quasi-biharmonic marginally trapped Lagrangian surfaces in nonflat Lorentzian complex space forms.

## 2 Preliminaries

### 2.1 Lagrangian submanifolds in complex space forms

Let  $\tilde{M}_s^n(4\epsilon)$  be a complex space form of complex dimension  $n$ , complex index  $s(\geq 0)$  and constant holomorphic sectional curvature  $4\epsilon$ . The complex index is defined as the complex dimension of the largest complex negative definite vector subspace of

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the tangent space. If  $s = 1$ , it is called *Lorentzian*. The curvature tensor  $\tilde{R}$  of  $\tilde{M}_s^n(4\epsilon)$  is given by

$$\begin{aligned}\tilde{R}(X, Y)Z &= \epsilon\{\langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle Z, JY \rangle JX \\ &\quad - \langle Z, JX \rangle JY + 2\langle X, JY \rangle JZ\},\end{aligned}\quad (2.1)$$

where  $\langle, \rangle$  and  $J$  are the metric tensor and the almost complex structure of  $\tilde{M}_s^n(4\epsilon)$  respectively.

Let  $\mathbf{C}_s^n$  be the  $n$ -dimensional complex space with complex coordinates  $z_1, \dots, z_n$ , endowed with the metric  $g_{n,s}(z, w) = \text{Re}(-\sum_{j=1}^s z_j \bar{w}_j + \sum_{i=s+1}^n z_i \bar{w}_i)$ . Put  $S_{2s}^{2n+1}(\epsilon) = \{z \in \mathbf{C}_s^n : g_{n+1,s}(z, z) = \frac{1}{\epsilon}\}$  for  $\epsilon > 0$  and  $H_{2s+1}^{2n+1}(\epsilon) = \{z \in \mathbf{C}_s^n : g_{n+1,s+1}(z, z) = \frac{1}{\epsilon}\}$  for  $\epsilon < 0$ .

The Hopf fibrations

$$\begin{aligned}\pi : S_{2s}^{2n+1}(\epsilon) &\rightarrow \mathbf{C}P_s^n(4\epsilon) : z \rightarrow z \cdot \mathbf{C}^*, \\ \pi : H_{2s+1}^{2n+1}(\epsilon) &\rightarrow \mathbf{C}H_s^n(4\epsilon) : z \rightarrow z \cdot \mathbf{C}^*,\end{aligned}$$

give  $\mathbf{C}P_s^n(4\epsilon)$  and  $\mathbf{C}H_s^n(4\epsilon)$  a unique pseudo-Riemannian metric of complex index  $s$  and curvature tensor (2.1) such that  $\pi$  is a pseudo-Riemannian submersion respectively.

Barros and Romero [2] showed that locally any complex space form  $\tilde{M}_s^n(4\epsilon)$  is isometric holomorphically to  $\mathbf{C}_s^n$ ,  $\mathbf{C}P_s^n(4\epsilon)$   $\mathbf{C}H_s^n(4\epsilon)$  according to  $\epsilon = 0$ ,  $\epsilon > 0$  or  $\epsilon < 0$ .

An  $n$ -dimensional submanifold  $M$  isometrically immersed in  $\tilde{M}_s^n(4\epsilon)$  is called *Lagrangian* if  $J$  interchanges the tangent and the normal spaces of  $M$ . For a Lagrangian submanifold  $M$  of complex space form  $\tilde{M}_s^n(4\epsilon)$ , we denote by  $\nabla$  and  $\tilde{\nabla}$  the Levi-Civita connections on  $M$  and  $\tilde{M}_s^n(4\epsilon)$ , respectively. The formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.2)$$

$$\tilde{\nabla}_X JY = -A_{JY}X + D_X JY, \quad (2.3)$$

for  $X, Y$  tangent to  $M$ , where  $h$ ,  $A$  and  $D$  are the second fundamental form, the shape operator and the normal connection. The mean curvature vector field is defined by  $H = \frac{1}{n}\text{tr}h$ . The shape operator and the second fundamental form are related by

$$\langle h(X, Y), JZ \rangle = \langle A_{JZ}X, Y \rangle \quad (2.4)$$

for  $X, Y, Z$  tangent to  $M$ . Since  $J$  is parallel, by (2.2) and (2.3) we have

$$D_X JY = J(\nabla_X Y), \quad (2.5)$$

$$A_{JY}X = -Jh(X, Y) = A_{JX}Y. \quad (2.6)$$

The equations of Gauss, Codazzi are given respectively by

$$\langle R(X, Y)Z, W \rangle = \epsilon(\langle \langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle) + \langle [A_{JZ}, A_{JW}](X), Y \rangle, \quad (2.7)$$

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z), \quad (2.8)$$

where  $X, Y, Z, W$  are vectors tangent to  $M$ ,  $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$  and  $\bar{\nabla}h$  is defined by

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \quad (2.9)$$

## 2.2 Legendre curves in the light cone

A vector  $X$  is called *spacelike* (resp. *timelike*) if it satisfies  $\langle X, X \rangle > 0$  (resp.  $\langle X, X \rangle < 0$ ). A vector  $X$  is called *lightlike* if it is nonzero and it satisfies  $\langle X, X \rangle = 0$ . The light cone  $\mathcal{LC} \subset \mathbf{C}_1^2$  is defined by  $\mathcal{LC} = \{z \in \mathbf{C}_s^n : \langle z, z \rangle = 0\}$ . A curve  $z(t)$  is called *null* if  $z'$  is lightlike for any  $t$ .

A curve  $z(t)$  in  $\mathcal{LC}$  is called *Legendre* if  $\langle z', iz \rangle = 0$  holds for any  $t$ . A Legendre curve  $z(t)$  in  $\mathcal{LC}$  is called *special Legendre* if  $\langle iz', z'' \rangle = 0$  holds for any  $t$ . For a unit speed special Legendre curve  $z(t)$  in  $\mathcal{LC}$ , the *squared Legendre curvature* and the *Legendre torsion* are defined by  $\hat{\kappa}^2 = \langle z', z' \rangle \langle z'', z'' \rangle$  and  $\hat{\tau} = \langle z', z' \rangle \langle z'', iz''' \rangle$ , respectively. For further details on Legendre curves in the light cone, see [4] and [6].

## 2.3 Bitension field

Let  $M$  and  $N$  be pseudo-Riemannian manifolds of dimension  $m$  and  $n$  respectively, and  $\phi : M \rightarrow N$  a smooth map. We denote by  $\nabla$  and  $\tilde{\nabla}$  the Levi-Civita connections on  $M$  and  $N$  respectively. Then the *tension field*  $\tau(\phi)$  is a section of the vector bundle  $\phi^*TN$  defined by

$$\tau(\phi) = \text{tr}(\nabla^\phi d\phi) = \sum_{i=1}^m \langle e_i, e_i \rangle \{ \nabla_{e_i}^\phi d\phi(e_i) - d\phi(\nabla_{e_i} e_i) \}.$$

Here  $\nabla^\phi$  and  $\{e_i\}$  denote the induced connection by  $\phi$  on the bundle  $\phi^*TN$ , which is the pull-back of  $\tilde{\nabla}$ , and a local orthonormal frame of  $M$ , respectively. If  $\phi$  is an isometric immersion, then  $\tau(\phi)$  and the mean curvature vector field  $H$  of  $M$  are related by

$$\tau(\phi) = mH. \quad (2.10)$$

A smooth map  $\phi$  is said to be a *harmonic map* if  $\tau(\phi) = 0$  at each point on  $M$ . If  $M$  and  $N$  are Riemannian manifolds, then  $\phi$  is harmonic if and only if it is a critical point of the *energy*

$$E(\phi) = \int_{\Omega} |d\phi|^2 dv_g$$

over every compactly supported region  $\Omega$  of  $M$ .

We define the *bitension field* as

$$\tau_2(\phi) = \sum_{i=1}^m \langle e_i, e_i \rangle \{ (\nabla_{e_i}^\phi \nabla_{e_i}^\phi - \nabla_{\nabla_{e_i} e_i}^\phi) \tau + R^N(\tau, d\phi(e_i)) d\phi(e_i) \}, \quad (2.11)$$

where  $R^N$  is the curvature tensor of  $N$ . If  $\phi$  is an isometric immersion and  $N$  is the complex space form  $\tilde{M}_s^n(4\epsilon)$ , then (2.1), (2.10) and (2.11) yield

$$\tau_2(\phi) = -m\Delta H + 5m\epsilon H, \quad (2.12)$$

where  $\Delta = -\sum_{i=1}^m \langle e_i, e_i \rangle (\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} - \tilde{\nabla}_{\nabla_{e_i} e_i})$ .

A smooth map  $\phi$  is called *biharmonic* if  $\tau_2(\phi) = 0$  at each point on  $M$ . Harmonic maps are clearly biharmonic. When  $M$  and  $N$  are Riemannian manifolds, the biharmonic map  $\phi$  is characterized as a critical point of the *bienergy*

$$E_2(\phi) = \int_{\Omega} |\tau(\phi)|^2 dv_g,$$

over every compactly supported region  $\Omega$  of  $M$ . For recent information on biharmonic maps between Riemannian manifolds, see [1] and [7].

A pseudo-Riemannian submanifold in a pseudo-Riemannian manifold isometrically immersed by  $\phi$  is called *marginally trapped* (or *quasi-minimal*) if the mean curvature vector field is lightlike, equivalently,  $\tau(\phi)$  is lightlike at each point on the submanifold. Analogously, we introduce the new class of submanifolds in pseudo-Riemannian manifolds as follows.

**Definition 1** A pseudo-Riemannian submanifold in a pseudo-Riemannian manifold isometrically immersed by  $\phi$  is called *quasi-biharmonic* if  $\tau_2(\phi)$  is lightlike at each point on the submanifold.

### 3 The bitension field of marginally trapped Lagrangian immersions

Let  $\phi : M \rightarrow \tilde{M}_1^2(4\epsilon)$  be a Lagrangian isometric immersion into a 2-dimensional Lorentzian complex space forms. Then, the real index of  $M$  is one. Choose a local orthonormal frame  $\{e_1, e_2\}$  such that  $\langle e_1, e_1 \rangle = 1$  and  $\langle e_2, e_2 \rangle = -1$ . Put  $\omega_i^j(e_k) = \langle \nabla_{e_k} e_i, e_j \rangle \langle e_j, e_j \rangle$  for  $i, j, k = 1, 2$ . Note that

$$\omega_1^2(e_k) = \omega_2^1(e_k), \quad \omega_1^1(e_k) = \omega_2^2(e_k) = 0. \quad (3.1)$$

It follows from (2.4) and (2.6) that the second fundamental form and the shape operator take the form

$$h(e_1, e_1) = aJe_1 + bJe_2, \quad (3.2)$$

$$h(e_1, e_2) = -bJe_1 + cJe_2, \quad (3.3)$$

$$h(e_2, e_2) = -cJe_1 + dJe_2, \quad (3.4)$$

$$A_{Je_1}e_1 = ae_1 + be_2, \quad (3.5)$$

$$A_{Je_1}e_2 = -be_1 + ce_2, \quad (3.6)$$

$$A_{Je_2}e_1 = -be_1 + ce_2, \quad (3.7)$$

$$A_{Je_2}e_2 = -ce_1 + de_2, \quad (3.8)$$

for some functions  $a, b, c, d$ .

We compute (2.9) using (3.2)-(3.4). In view of (3.1) we get

$$\begin{aligned} (\bar{\nabla}_{e_2}h)(e_1, e_1) &= (e_2a + 3b\omega_1^2(e_2))Je_1 \\ &\quad + (e_2b + (a - 2c)\omega_1^2(e_2))Je_2, \end{aligned} \quad (3.9)$$

$$\begin{aligned} (\bar{\nabla}_{e_1}h)(e_1, e_2) &= -(e_1b + (a - 2c)\omega_1^2(e_1))Je_1 \\ &\quad + (e_1c - (2b + d)\omega_1^2(e_1))Je_2, \end{aligned} \quad (3.10)$$

$$\begin{aligned} (\bar{\nabla}_{e_1}h)(e_2, e_2) &= (-e_1c + (2b + d)\omega_1^2(e_1))Je_1 \\ &\quad + (e_1d - 3c\omega_1^2(e_1))Je_2, \end{aligned} \quad (3.11)$$

$$\begin{aligned} (\bar{\nabla}_{e_2}h)(e_1, e_2) &= -(e_2b + (a - 2c)\omega_1^2(e_2))Je_1 \\ &\quad + (e_2c - (2b + d)\omega_1^2(e_2))Je_2. \end{aligned} \quad (3.12)$$

From (2.8) and (3.9)-(3.12) we obtain

$$e_2a + 3b\omega_1^2(e_2) = -e_1b - (a - 2c)\omega_1^2(e_1), \quad (3.13)$$

$$e_2b + (a - 2c)\omega_1^2(e_2) = e_1c - (2b + d)\omega_1^2(e_1), \quad (3.14)$$

$$e_1d - 3c\omega_1^2(e_1) = e_2c - (2b + d)\omega_1^2(e_2). \quad (3.15)$$

Denote the Gauss curvature of  $M$  by  $G$ . Then, the Gauss equation (2.7) implies that  $G$  satisfies

$$G = ac + b^2 + bd - c^2 + \epsilon. \quad (3.16)$$

From now on, let us assume that  $\phi : M \rightarrow \tilde{M}_1^2(4\epsilon)$  is a marginally trapped Lagrangian immersion into a 2-dimensional Lorentzian complex space forms and  $\{e_1, e_2\}$  is an orthonormal frame on  $M$  satisfying  $\langle e_1, e_1 \rangle = 1$ ,  $\langle e_2, e_2 \rangle = -1$ ,

$$H = \alpha(Je_1 + Je_2) \quad (3.17)$$

for some function  $\alpha$  which is nowhere zero, and the second fundamental form is expressed as (3.2) – (3.4).

The main purpose of this section is to express the bitension of  $\phi$  by  $a, b, c, \alpha, e_1, e_2$  and  $\omega_i^j(e_k)$ .

By (3.2), (3.4) and (3.17) we have

$$2\alpha = a + c = b - d. \quad (3.18)$$

Combining (3.16) and (3.18) yields

$$G = (a - 2b - c)(c - b) + \epsilon. \quad (3.19)$$

In view of (2.12), we need the following basic formula, which is obtained by a straightforward computation using (2.2) and (2.3) (cf. (4.7) in p.270 of [3]):

$$\Delta H = \Delta^D H + \sum_{i=1}^2 \langle e_i, e_i \rangle h(e_i, A_H e_i) + \sum_{i=1}^2 \langle e_i, e_i \rangle (A_{D_{e_i} H} e_i + (\nabla_{e_i} A_H) e_i), \quad (3.20)$$

where  $\Delta^D = -\sum_{i=1}^2 \langle e_i, e_i \rangle (D_{e_i} D_{e_i} - D_{\nabla_{e_i} e_i})$ .

First, we have

**Lemma 2** *Let  $M$  be a marginally trapped Lagrangian surface in  $\tilde{M}_1^2(4\epsilon)$ . Then, the normal part of  $\Delta H$  is expressed as*

$$\begin{aligned} (\Delta H)^\perp &= \alpha\{(a - 2b - c)(a + 2b - c) - \epsilon\}Je_1 \\ &\quad + \alpha\{(a - 2b - c)(-a + 2b - 3c) - \epsilon\}Je_2. \end{aligned} \quad (3.21)$$

*Proof:* By Proposition 3 in [8], we have

$$\Delta^D H = -GH. \quad (3.22)$$

( In [8], the relation  $\Delta^D H = GH$  was derived. But the sign of the Gauss curvature (3.16) in [8] was incorrect.) Using (3.2)-(3.4) we get

$$\sum_{i=1}^2 \langle e_i, e_i \rangle h(e_i, A_H e_i) = \alpha(a - 2b - c)\{(a + b)Je_1 + (-a + b - 2c)Je_2\}. \quad (3.23)$$

From (3.19), (3.20) (3.22) and (3.23) we obtain (3.21). ■

Next we compute the tangential part of  $\Delta H$ . To do so, we need the following Lemma.

**Lemma 3** *Let  $M$  be a marginally trapped Lagrangian surface in  $\tilde{M}_1^2(4\epsilon)$ . Then, we have*

$$e_1\alpha + e_2\alpha = -\alpha(\omega_1^2(e_1) + \omega_1^2(e_2)), \quad (3.24)$$

$$e_1a - e_1b + e_2b + e_2c = (a - 3b - 2c)\omega_1^2(e_1) + (-2a + 3b + c)\omega_1^2(e_2), \quad (3.25)$$

$$e_2a + e_1b - e_2b + e_1c = (-2a + 3b + c)\omega_1^2(e_1) + (a - 3b - 2c)\omega_1^2(e_2). \quad (3.26)$$

*Proof:* Combining (3.13), (3.15) and (3.18) shows (3.24). Similarly, by (3.14), (3.15) and (3.18) we get (3.25). We obtain (3.26) from (3.24), (3.25) and (3.18) immediately. ■

**Lemma 4** *Let  $M$  be a marginally trapped Lagrangian surface in  $\tilde{M}_1^2(4\epsilon)$ . Then, the tangential part of  $\Delta H$  is expressed as*

$$(\Delta H)^\top = 2(a - 2b - c)\{e_1\alpha + \alpha\omega_1^2(e_1)\}(e_1 - e_2). \quad (3.27)$$

*Proof:* It follows from (3.5)-(3.8) and (3.18) that

$$A_H e_1 = \alpha\{(a - b)e_1 + (b + c)e_2\}, \quad (3.28)$$

$$A_H e_2 = \alpha\{-(b + c)e_1 + (b - a)e_2\}. \quad (3.29)$$

By (3.1) (3.28) and (3.29) we get

$$\begin{aligned} \nabla_{e_1}(A_H e_1) &= (e_1\alpha)\{(a - b)e_1 + (b + c)e_2\} \\ &\quad + \alpha\{(e_1a - e_1b)e_1 + (e_1b + e_1c)e_2 \\ &\quad + (a - b)\omega_1^2(e_1)e_2 + (b + c)\omega_1^2(e_1)e_1\}, \end{aligned} \quad (3.30)$$

$$A_H(\nabla_{e_1} e_1) = \alpha\omega_1^2(e_1)\{-(b + c)e_1 + (b - a)e_2\}, \quad (3.31)$$

$$\begin{aligned} \nabla_{e_2}(A_H e_2) &= (e_2\alpha)\{-(b + c)e_1 + (b - a)e_2\} \\ &\quad + \alpha\{(-e_2b - e_2c)e_1 + (e_2b - e_2a)e_2\} \\ &\quad + \alpha\{-(b + c)\omega_1^2(e_2)e_2 + (b - a)\omega_1^2(e_2)e_1\}, \end{aligned} \quad (3.32)$$

$$A_H(\nabla_{e_2} e_2) = \alpha(\omega_1^2(e_2))\{(a - b)e_1 + (b + c)e_2\}. \quad (3.33)$$

Using (2.5), (3.1) and (3.17) we obtain

$$D_{e_1} H = \{e_1\alpha + \alpha\omega_1^2(e_1)\}(Je_1 + Je_2),$$

$$D_{e_2} H = \{e_2\alpha + \alpha\omega_1^2(e_2)\}(Je_1 + Je_2),$$

which imply that

$$A_{D_{e_1} H} e_1 = \{e_1\alpha + \alpha\omega_1^2(e_1)\}\{(a - b)e_1 + (b + c)e_2\}, \quad (3.34)$$

$$A_{D_{e_2} H} e_2 = \{e_2\alpha + \alpha\omega_1^2(e_2)\}\{-(b + c)e_1 + (b - a)e_2\}. \quad (3.35)$$

By a straightforward computation using (3.20), (3.30)-(3.35) we find that the tangential part  $(\Delta H)^\top$  of  $\Delta H$  satisfies

$$\begin{aligned}\langle (\Delta H)^\top, e_1 \rangle &= (e_1\alpha)(a-b) + (e_2\alpha)(b+c) + \alpha(e_1a - e_1b + e_2b + e_2c) \\ &\quad + 2\alpha(b+c)\omega_1^2(e_1) + 2\alpha(a-b)\omega_1^2(e_2) \\ &\quad + \{e_1\alpha + \alpha\omega_1^2(e_1)\}(a-b) + \{e_2\alpha + \alpha\omega_1^2(e_2)\}(b+c), \quad (3.36) \\ -\langle (\Delta H)^\top, e_2 \rangle &= (e_1\alpha)(b+c) - (e_2\alpha)(b-a) + \alpha(e_2a + e_1b - e_2b + e_1c) \\ &\quad + 2\alpha(a-b)\omega_1^2(e_1) + 2\alpha(b+c)\omega_1^2(e_2) \\ &\quad + \{e_1\alpha + \alpha\omega_1^2(e_1)\}(b+c) - \{e_2\alpha + \alpha\omega_1^2(e_2)\}(b-a). \quad (3.37)\end{aligned}$$

Substituting (3.24)-(3.26) into (3.36) and (3.37) gives us (3.27). ■

By (2.12), Lemma 2 and Lemma 4, we obtain the following.

**Lemma 5** *Let  $\phi : M \rightarrow \tilde{M}_1^2(4\epsilon)$  be a marginally trapped Lagrangian immersion. The bitension field of  $\phi$  is expressed as*

$$\begin{aligned}\tau_2(\phi) &= -4(a-2b-c)\{e_1\alpha + \alpha\omega_1^2(e_1)\}(e_1 - e_2) \\ &\quad - 2\alpha\{(a-2b-c)(a+2b-c) - 6\epsilon\}Je_1 \\ &\quad - 2\alpha\{(a-2b-c)(-a+2b-3c) - 6\epsilon\}Je_2. \quad (3.38)\end{aligned}$$

## 4 Main results

Recently, the following result has been obtained by the author.

**Theorem 6 ([9])** *Let  $M$  be a biharmonic marginally trapped Lagrangian surface in a 2-dimensional Lorentzian complex space form of constant holomorphic sectional curvature  $4\epsilon$ . Then  $\epsilon = 0$ , that is, the ambient space is  $\mathbf{C}_1^2$ . Moreover,  $M$  is locally congruent to*

$$\phi(x, y) = c_1 x e^{if(y)} + z(y),$$

where  $f(y)$  is a real-valued function,  $c_1$  is a lightlike vector,  $z(y)$  is a null curve in  $\mathbf{C}_1^2$  satisfying  $\langle iz', c_1 e^{if(y)} \rangle = 0$  and  $\langle z', c_1 e^{if(y)} \rangle = -1$ .

In this section, we classify quasi-biharmonic marginally trapped Lagrangian surfaces in 2-dimensional Lorentzian complex space forms. Very unlike the biharmonic case, there exist a lot of quasi-biharmonic marginally trapped Lagrangian surfaces in 2-dimensional *nonflat* Lorentzian complex space forms.

In the case when the ambient space is flat, we have

**Theorem 7** *Let  $M$  be a quasi-biharmonic marginally trapped Lagrangian surface in  $\mathbf{C}_1^2$ . Then,  $M$  is locally congruent to*

$$\phi(x, y) = e^{i\mu y} z(x), \quad (4.1)$$

where  $\mu$  is a nonzero real number and  $z(x)$  is a null curve in the light cone  $\mathcal{LC}$  satisfying

$$\langle z, iz' \rangle = \mu^{-1}, \quad z'' \neq 0 \quad (4.2)$$

at each point on  $M$ .

*Proof:* Let  $\phi : M \rightarrow \mathbf{C}_1^2$  be a marginally trapped Lagrangian immersion and let  $\{e_1, e_2\}$  be an orthonormal frame on  $M$  satisfying  $\langle e_1, e_1 \rangle = 1$ ,  $\langle e_2, e_2 \rangle = -1$  and (3.17). Suppose that the second fundamental form is given by (3.2)-(3.3).

If  $M$  is quasi-biharmonic, that is,  $\tau_2(\phi)$  is lightlike, then from (3.38) we obtain that  $a - 2b - c \neq 0$  and  $b = c$ . Hence, we get  $G = 0$  by (3.19). Thus, there exists a local coordinate system  $\{s, t\}$  such that the metric tensor of  $M$  is given by

$$g = ds^2 - dt^2. \quad (4.3)$$

We may assume that  $e_1 = \partial_s$  and  $e_2 = \partial_t$ . We put  $\partial_x = \frac{1}{\sqrt{2}}(\partial_s - \partial_t)$  and  $\partial_y = -\frac{1}{\sqrt{2}}(\partial_s + \partial_t)$ . Then, by (4.3) the metric tensor  $g$  is expressed as

$$g = -dx dy. \quad (4.4)$$

Moreover, a straightforward computation using (3.2)-(3.4) shows

$$\begin{aligned} h(\partial_x, \partial_x) &= \frac{a-3b}{\sqrt{2}} J \partial_y, \\ h(\partial_x, \partial_y) &= \sqrt{2} \alpha J \partial_x, \\ h(\partial_y, \partial_y) &= \sqrt{2} \alpha J \partial_y. \end{aligned} \quad (4.5)$$

Put  $\lambda = \frac{a-3b}{\sqrt{2}}$  and  $\mu = \sqrt{2} \alpha$ . Note that  $\lambda \neq 0$  and  $\mu \neq 0$  at each point of  $M$ . By (2.1), (4.4) and (4.5), we see that the Lagrangian immersion  $\phi(x, y)$  satisfies

$$\phi_{xx} = i\lambda \phi_y, \quad \phi_{xy} = i\mu \phi_x, \quad \phi_{yy} = i\mu \phi_y. \quad (4.6)$$

The compatibility condition of (4.6) is given by

$$\lambda_y = \mu_x = 0, \quad \mu_y = 0. \quad (4.7)$$

From (4.7) we obtain that  $\lambda = \lambda(x)$  and  $\mu$  is a nonzero constant.

By solving the last two equations of (4.6), we have

$$f(x, y) = e^{i\mu y} z(x) \quad (4.8)$$

for some  $\mathbf{C}_1^2$ -valued function  $z$ . Substituting (4.8) into the first equation of (4.6) gives

$$z'' = -\mu \lambda z. \quad (4.9)$$

It follows from (4.4) that

$$\langle z, z \rangle = \langle z', z' \rangle = 0, \quad \langle z, iz' \rangle = \mu^{-1}. \quad (4.10)$$

Hence,  $\phi$  is expressed as (4.1) satisfying (4.2). Note that (4.9) is a consequence of (4.10).

Conversely, we can check that the immersion given in Theorem 7 is a quasi-biharmonic marginally trapped Lagrangian immersion into  $\mathbf{C}_1^2$ . The proof is finished. ■

In the case when the ambient space is nonflat, we have



**Theorem 8** *Let  $M$  be a marginally trapped Lagrangian surface in a 2-dimensional Lorentzian complex space form of constant holomorphic sectional curvature  $4\epsilon \neq 0$ . Then,  $M$  is quasi-biharmonic if and only if the Gauss curvature of  $M$  is equal to  $\epsilon$ .*

*Proof:* Let  $\phi : M \rightarrow \tilde{M}_1^2(4\epsilon)$  be a marginally trapped Lagrangian immersion and let  $\{e_1, e_2\}$  be an orthonormal frame on  $M$  satisfying  $\langle e_1, e_1 \rangle = 1$ ,  $\langle e_2, e_2 \rangle = -1$  and (3.17). Suppose that the second fundamental form is given by (3.2)-(3.3).

Assume that  $\epsilon \neq 0$  and  $M$  is quasi-biharmonic. Then, it follows from (3.38) that  $a - 2b - c = 0$  or  $(b - c)(a - 2b - c) = 3\epsilon$ . If  $(b - c)(a - 2b - c) = 3\epsilon$  holds, then by using (3.19) we get  $G = 2\epsilon$ . On the other hand, by virtue of Theorem 5.1 and 6.1 in [6], we know that  $G = 0$  or  $\epsilon$ . Therefore, we obtain  $\epsilon = 0$ , however this contradicts the assumption. Accordingly, we have  $a - 2b - c = 0$ . This and (3.19) show  $G = \epsilon$ .

Conversely, assume that  $G = \epsilon \neq 0$ . Then, by (3.19) we have that  $b = c$  or  $a - 2b - c = 0$ . If  $b = c$  holds, by combining (3.13), (3.14), (3.18) and (3.24) we get

$$\omega_1^2(e_1) = -\frac{e_1\alpha}{\alpha}, \quad \omega_1^2(e_2) = -\frac{e_2\alpha}{\alpha}. \quad (4.11)$$

On the other hand, the Gauss curvature  $G$  is given by

$$G = -e_1(\omega_1^2(e_2)) + e_2(\omega_1^2(e_1)) + (\omega_1^2(e_1))^2 - (\omega_1^2(e_2))^2. \quad (4.12)$$

Substituting (4.11) into (4.12), we have

$$\begin{aligned} G &= \frac{1}{\alpha} \{e_1 e_2 \alpha - e_2 e_1 \alpha\} + \frac{1}{\alpha^2} \{(e_1 \alpha)^2 - (e_2 \alpha)^2\} \\ &= \frac{1}{\alpha} [e_1, e_2] \alpha + \frac{1}{\alpha^2} \{(e_1 \alpha)^2 - (e_2 \alpha)^2\} \\ &= \frac{1}{\alpha} \{\omega_1^2(e_1) e_1 \alpha - \omega_1^2(e_2) e_2 \alpha\} + \frac{1}{\alpha^2} \{(e_1 \alpha)^2 - (e_2 \alpha)^2\} \\ &= 0, \end{aligned}$$

which contradicts the assumption. Hence, we have  $a - 2b - c = 0$ . Thus, it follows from (3.38) that  $\tau_2(\phi)$  is lightlike, that is,  $M$  is quasi-biharmonic because of  $\epsilon \neq 0$ . This completes the proof. ■

By combining Theorem 5.1, 6.1 in [6] and Theorem 8, we obtain

**Theorem 9** *A quasi-biharmonic marginally trapped Lagrangian surface in  $\mathbf{CP}_1^2(4)$  is locally congruent to the composition  $\pi \circ L$ , where  $\pi : S_2^5(1) \rightarrow \mathbf{CP}_1^2(4)$  is the Hopf fibration and  $L$  is one of the following four families:*

(i)

$$L(x, y) = \frac{1}{a(x+y)} \left( e^{\sqrt{2}ia y} (\sqrt{2} + ia(x-y)), e^{\sqrt{2}ia y} a(x-y), \sqrt{2} + ia(x+y) \right),$$

where  $a$  is a nonzero real number.

(ii)

$$L(x, y) = \left( \frac{2}{x+y} + \sqrt{2}if(y) \right) z(y) - z'(y),$$

where  $f(y)$  is a nonconstant real-valued function and  $z(y)$  is a unit speed spacelike Legendre curve with the squared Legendre curvature  $\hat{\kappa}^2 = 6f(y)^2$  in the light cone  $\mathcal{LC}$ .

(iii)

$$L(x, y) = \frac{e^{\frac{i}{\sqrt{2}}bx}}{3by} \left( (\sqrt{6} + i\sqrt{3}by) \cosh\left(\frac{\sqrt{6}}{2}bx\right) - 3(\sqrt{2}i + by) \sinh\left(\frac{\sqrt{6}}{2}bx\right), \right. \\ \left. 3by \cosh\left(\frac{\sqrt{6}}{2}bx\right) + \sqrt{3}(iby - 2\sqrt{2}) \sinh\left(\frac{\sqrt{6}}{2}bx\right), \frac{\sqrt{6} + i\sqrt{3}by}{e^{3ibx/\sqrt{2}}} \right),$$

where  $b$  is a positive real number.

(iv)

$$L(x, y) = \frac{z(y)}{x + y} - \frac{z'(y)}{2},$$

where  $z(y)$  is a unit speed timelike special Legendre curve in the light cone  $\mathcal{LC}$  with zero squared Legendre curvature and nonzero Legendre torsion.

**Theorem 10** A quasi-biharmonic marginally trapped Lagrangian surface in  $\mathbf{CH}_1^2(-4)$  is locally congruent to the composition  $\pi \circ L$ , where  $\pi : H_3^5(-1) \rightarrow \mathbf{CH}_1^2(-4)$  is the Hopf fibration and  $L$  is one of the following four families:

(i)

$$L(x, y) = \frac{1}{a(x - y)} \left( ae^{\sqrt{2}iax}(x + y), a(x - y) + i\sqrt{2}, e^{\sqrt{2}iax}(a(x + y) + \sqrt{2}i) \right),$$

where  $a$  is a nonzero real number.

(ii)

$$L(x, y) = \left( \frac{2}{x - y} - \sqrt{2}if(y) \right) z(y) + z'(y),$$

where  $f(y)$  is a nonconstant real-valued function and  $z(y)$  is a unit speed timelike Legendre curve with nonconstant squared Legendre curvature in the light cone  $\mathcal{LC}$ .

(iii)

$$L(x, y) = \frac{e^{\frac{-i}{\sqrt{2}}bx}}{3by} \left( 3(by + i\sqrt{2}) \cosh\left(\frac{\sqrt{6}}{2}bx\right) + 3(\sqrt{2} + iby) \sinh\left(\frac{\sqrt{6}}{2}bx\right), \right. \\ \left. e^{\frac{3}{\sqrt{2}}ibx}(\sqrt{6} + i\sqrt{3}by), 3by \sinh\left(\frac{\sqrt{6}}{2}bx\right) - \sqrt{3}(iby - 2\sqrt{2}) \cosh\left(\frac{\sqrt{6}}{2}bx\right) \right),$$

where  $b$  is a nonzero real number.

(iv)

$$L(x, y) = \frac{z(y)}{x - y} - \frac{z'(y)}{2},$$

where  $z(y)$  is a unit speed spacelike special Legendre curve in the light cone  $\mathcal{LC}$  with zero squared Legendre curvature and nonzero Legendre torsion.

## 5 Remarks (added on December 2, 2014)

**Remark 11** By changing the sign of the metric of  $\mathbf{CP}_1^2(4)$ , we have that  $\mathbf{CP}_1^2(4)$  is holomorphically anti-isometric to  $\mathbf{CH}_1^2(-4)$ . Hence, it is sufficient to consider only the case of  $\mathbf{CP}_1^2(4)$ .

**Remark 12** Comments on Theorem 5.1 in [6]: Surfaces in Case (A.b.ii) satisfy  $f = 0$ . Hence, by (5.17) we have  $\mu = 0$ . This and (5.8) imply that these surfaces are minimal. Accordingly, (p.1.4) and (p.1.5) in Theorem 5.1 should be removed from the list of marginally trapped surfaces. Also, surfaces in Case (A.b.i) are represented by (5.30) with (5.31), where  $\delta = 0$  and  $f = a$  for some nonzero constant  $a$ .

Considering these facts, Theorem 9 of this paper should be read as follows.

**Theorem 9** *A quasi-biharmonic marginally trapped Lagrangian surface in  $\mathbf{CP}_1^2(4)$  is locally congruent to the composition  $\pi \circ L$ , where  $\pi : S_2^5(1) \rightarrow \mathbf{CP}_1^2(4)$  is the Hopf fibration and  $L$  is given by*

$$L(x, y) = \left( \frac{2}{x + y} + \sqrt{2}i f(y) \right) z(y) - z'(y).$$

Here  $f(y)$  is a real-valued function and  $z(y)$  is a unit speed spacelike Legendre curve in the light cone  $\mathcal{LC}$  which satisfies

$$z''' = 2\sqrt{2}i f z'' + 2(f^2 + \sqrt{2}i f') z' + (\sqrt{2}i(f'' + 2\delta) + 2f f') z$$

for some real-valued function  $\delta(y)$ .

The squared Legendre curvature and the Legendre torsion of  $z$  described in Theorem 9 are given by  $6f^2$  and  $-8\sqrt{2}f^3 + \sqrt{2}(f'' + 2\delta)$ , respectively (see the proof of Case (A.b.iii) in [6]).

**Remark 13** In [”An isometric embedding of the complex hyperbolic space in a pseudo-Euclidean space and its application to the study of real hypersurfaces”, Tsukuba J. Math. **14** (1990), 293-313], Garay and Romero constructed an isometric embedding  $\Psi$  of  $\mathbf{CH}^n(-4)$  into some pseudo-Euclidean space, which is provided with good geometric properties as the first standard embedding of  $\mathbf{CP}^n(4)$ .

Let  $\phi : M \rightarrow \mathbf{CH}^n(-4)$  be a real hypersurface with constant mean curvature. Then, they showed that  $\tau_2(\Psi \circ \phi) = Q$  holds for some non-zero constant vector if and only if  $M$  is locally congruent to the horosphere  $\pi(\{(z_1, z_2, \dots, z_{n+1}) \in H_1^{2n+1}(-1) : |z_1 - z_2|^2 = 1\})$ . In this case,  $Q$  is lightlike, i.e., the isometric immersion  $\Psi \circ \phi$  is quasi-biharmonic.

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